博士資格考試量子力學累積考題

1. Describe:
   (1) uncertainty principle,
   (2) probability current density,
   (3) Pauli's exclusion principle,
   (4) zero point energy,
   (5) Fermi's golden rule,
   (6) stark effect,
   (7) Bose-Einstein and Fermi-Dirac distributions, and
   (8) the origins of the fine structure and hyperfine splitting.

2. Describe the differences between Schrödinger's and Heisenberg's views of quantum mechanics.

3. (a) Find eigenenergies and corresponding eigenfunctions of a particle in the potential well given by $V(x) = 0 \quad if \quad |x| < a \quad and \quad V(x) = \infty \quad if \quad |x| > a.$

   (b) Find coordinate and the momentum matrices within the complete basis set of eigenfunctions found in (a).

4. Show that if hermitian operators **A** and **B** satisfy the commutation relation [**A**,**B**]=i**C** the following relation holds:

$$\sqrt{\overline{(\Delta A)^2} \quad \overline{(\Delta B)^2}} \geq \frac{|\overline{C}|}{2}.$$

5. Prove if two operators **A** and **B** commute, they share common eigenstates.

6. Find energy levels and corresponding wave functions of the stationary states of a two-particle planar rotator with a moment of inertia equal to I=μa^2, where μ is the reduced mass of this pair of particles and a is their separation.

7. Two particles of mass m are attached to the ends of a massless rigid rod of length a. The system is free to rotate in three dimensions about the fixed center. (a) Find eigenenergies of this rigid rotor. (b) What is the degeneracy of the nth energy level?

8. A system described by the Hamiltonian

$$H = -\frac{\hbar^2}{2m}\nabla^2 + \frac{m}{2}(\omega_1^2 x^2 + \omega_2^2 y^2 + \omega_3^2 z^2)$$

   is called an anisotropic harmonic oscillator. Determine eigenenergies of this system, and for the isotropic case ($\omega_1=\omega_2=\omega_3=\omega$) calculate the degeneracy of the level $E_n = \hbar\omega(n+3/2)$, where n is a positive integer.

9. Find approximate eigenenergies of an anharmonic oscillator whose Hamiltonian is given by

$$H = H_0 + H_1 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2 + ax^3 + bx^4,$$

where a and b are small constants.

10. (a) Find bound-state eigenenergies of a particle in an attractive potential $V(x) = -aV_0\delta(x)$.

(b) Find the probability of transmission of a particle moving along the x-axis through a delta-function potential barrier at the origin $V(x) = aV_0\delta(x)$.

11. (a) Determine eigenenergies and corresponding normalized eigenfunctions of a particle in a potential well described by

$$V(x) = \begin{cases} \infty & x < 0 \\ 0 & 0 < x < a \\ \infty & x > a \end{cases}$$

(b) Calculate the expectation values of x and p, and uncertainties Δx and Δp for the ground state.

(c) Explain that the ground-state energy of this particle is different from zero.

12. Suppose we have two non-interacting particles, both of mass m, in the previous infinite square well. Find the ground-state and first excited state eigenenergies and corresponding eigenfunctions of this pair of particles

(a) if the two particles are identical bosons, and

(b) if they are identical fermions.

13. A particle is in the ground state in a box with sides at x=0 ad $x=a$. Suddenly the walls of the box are moved to $\pm\infty$, so that the particle is free. What is the probability that the particle has a momentum in the range $(p, p+dp)$?

14. Let $\mathbf{s}_1$ and $\mathbf{s}_2$ be the spin operators of two spin-1/2 particles. Find the simultaneous eigenstates of operators $\mathbf{s}^2$ and $\mathbf{s}_z$, where $\mathbf{s}=\mathbf{s}_1+\mathbf{s}_2$. Show that these eigenstates are also eigenstates of the operators $\mathbf{s}_1 \cdot \mathbf{s}_2$.

15. Find eigenenergies of a particle in the spherical potential well given by

$$V(x) = \begin{cases} -V_0 & r < a \\ 0 & r > a \end{cases}$$

16. Assume that, at time t=0, the wave function of a free particle is of the form

$$\psi(x,0)=\frac{1}{(\pi\Delta^2)^{1/4}}\exp\left(\frac{-x^2}{2\Delta^2}\right), \text{ find } \Psi(x,t).$$

17. Consider a particle subject to a constant force F in one dimension. Solve for the propagator in coordinate space.
18. A plane rigid rotator having a moment of inertia $\boldsymbol{I}$ and an electric dipole moment d is placed in a uniform electric field $\boldsymbol{E}$. By considering the electric field as a perturbation, determine the first non-vanishing correction to the energy levels of the rotator.
19. A charged-particle linear harmonic oscillator is in a time-dependent uniform electric field given by $\varepsilon(t)=\frac{A}{\pi\tau}e^{-(t/\tau)^2}$, where A and τ are constants. If at $t=-\infty$, the oscillator is in its ground state, find, to the first order approximation, the probability that it will be in its first excited state at $t=\infty$.
20. Consider a three-dimensional infinite cubical well described by $V(x,y,z)=0$ for $0<x<a$, $0<y<a$ and $0<z<a$ and $V(x,y,z)=\infty$ otherwise.
    (a) Find the stationary states.
    (b) Find the ground state and the first excited states and their degeneracy.
    Now let's introduce the perturbation $H_1=V_0$ for $0<x<a/2$ and $0<y<a/2$ and $H_1=0$ otherwise.

    Find the first-order corrections to (c) the ground state and (d) the first excited state energies.

21. Use a Gaussian trial function to obtain the upper bound of the ground-state energy of an one-dimensional harmonic oscillator.

22. Find the total cross-section of the scattering of slow particles by the spherical potential well $V(r)=-V_0 \; if \quad r<r_0 \quad and \quad V(r)=0 \quad if \quad r>r_0$.

23. Write down the Hamiltonian of a charged particle moving in the electromagnetic field.

24. A charged particle with mass m is constrained to move on a spherical shell with a radius of R in a weak uniform electric field E. Obtain corrections to the eigenenergies to the second order in the field strength.

25. The eigenvalue equation of the harmonic oscillator is given as:

$$(\frac{\hat{p}^2}{2m}+\frac{1}{2}m\omega^2\hat{x}^2)\,|\,E\rangle = E\,|\,E\rangle .$$

We usually describe eigenstates in terms of Hermite polynomials. Formulate this problem in number space, i.e. describe the eigenstates as $|n\rangle$, instead using the following lowering and raising operators:

$$\hat{a} = (\frac{m\omega}{2\hbar})^{1/2}\hat{x} + i(\frac{1}{2m\omega\hbar})^{1/2}\hat{p} \quad \text{and} \quad \hat{a}^{+} = (\frac{m\omega}{2\hbar})^{1/2}\hat{x} - i(\frac{1}{2m\omega\hbar})^{1/2}\hat{p}$$

(a) Find $\hat{a}|n\rangle$ and $\hat{a}^{+}|n\rangle$.

(b) Calculate expectation values $\langle 0|\hat{p}^2|0\rangle$ and $\langle 3|\hat{x}^2|2\rangle$.

26. (A) Solve the time-independent Schrodinger equation with the time-independent perturbation method. Find the first- and second-order corrections to the eigenenergy and the first-order correction to the eigenfunction. If the unperturbed states are two-fold degenerate, find out the first-order corrections to eigenenergy.

(B) Consider a quantum system with just three linearly independent states.

The Hamiltonian, in matrix form, is given by

$$\hat{H} = V_0 \begin{pmatrix} 1-\varepsilon & 0 & 0 \\ 0 & 1 & \varepsilon \\ 0 & \varepsilon & 2 \end{pmatrix},$$

where V_0 is a constant and ε is a small number.

(a) Write down eigenvalues and corresponding eigenvectors of the unperturbed Hamiltonian(ε=0).

(b) Solve for exact eigenvalues of $\hat{H}$. Expand each of them as a power series in ε up to the second order.

(c) Use first- and second-order nondegenerate perturbation theory to find the approximate eigenvalue of the nondegenerate state of unperturbed $\hat{H}$. Compare it with the exact result from (b).

(d) Use degenerate perturbation theory to find the first-order correction to the two degenerate states of the unperturbed $\hat{H}$. (Compare them with the exact results).

27. (A) For a system with a spherically symmetric potential like the hydrogen atom, states are specified by quantum numbers n, ℓ, and m. What are the selection rules for spontaneous emission?

(B) Derive these selection rules using the properties of spherical harmonic functions.

28. At time t=0 a spin 1/2 particle with spin in the *x*-direction enters a region of space in which there is a uniform magnetic field *B* in the z-direction. Find the probability that at time *t* the spin is still in the *x*-direction.

29. (a) If $\varphi_i(x)$ and $\varphi_j(x)$ are two different non-degenerate eigenfunctions of the time-independent Schrödinger equation for a potential V(x). Show that

$$\int_{-\infty}^{\infty} \varphi_i^*(x)\varphi_j(x)dx = 0$$

(b) Write down the normalized eigenfunction (including orbital and spin parts) of the ground state of a lithium atom

30. Consider a hard-sphere central potential given by

$$V(r) = \begin{cases} \infty & for \quad r < R \\ 0 & for \quad r > R \end{cases}.$$

Using the method of partial waves to calculate the total scattering cross section of slow incident particles.

31. Show that for any normalized function $|\Psi\rangle$, $\langle\Psi|\hat{H}|\Psi\rangle \geq E_0$,where E_0 is the ground-state energy (i.e. the lowest eigenvalue). And show that if $|\delta\Psi\rangle$ is a small deviation from the ground-state eigenfunction $|\Psi_0\rangle$, i.e. $|\Psi\rangle = |\Psi_0\rangle + |\delta\Psi\rangle$ the lowest order of the deviation of the energy $E = \dfrac{\langle\Psi|\hat{H}|\Psi\rangle}{\langle\Psi|\Psi\rangle}$ from E_0 is $(\delta\Psi)^2$.

32. If $|n\rangle$ with *n*=0,1,2,3, .., are the eigenstates of the number operator $\hat{N} = \hat{a}^+\hat{a}$ of a one-dimensional simple harmonic oscillator, calculate the matrices of the position operator $\hat{x}$ and the momentum operator $\hat{p}$ based on the complete basis set of $\{|n\rangle\}$.

33. Find the uncertainty relation between ϕ, the rotation angle about the z-axis, and $\mathbf{L_z}$, the z component of the angular momentum.

34. A particle is in a potential $V(x) = V_0 \sin(2\pi x / a)$, which is invariant under the transformation $x \rightarrow x+ma$, where m is an integer. Is its momentum conserved? Discuss eigenvalues and corresponding eigenfunctions of the Hamiltonian of this particle.

35. Let R(ϕ,**n**) be the operator that rotates a vector by ϕ about the axis **n**. Show that the four successive infinitesimal rotations, R(ε_x,**i**),R(ε_y,**j**),R(-ε_x,**i**), and lastly R(-ε_y,**j**) is equivalent to R(-ε_x ε_y,**k**). Then use this identity to show that [$\mathbf{L}_X$,$\mathbf{L}_Y$]=iħ$\mathbf{L}_Z$. (This is called consistent test).

36. Consider a particle in a state described by $\Psi(x,y,z) = N(x+y+2z)\exp(-\alpha r)$, where N is a normalization factor. Show that the probabilities of finding the eigenstates of $\mathbf{L}_Z$ with eigenvalue=$m\hbar$ are P(m=0)=2/3, P(m=+1)=1/6, and P(m= –1)=1/6.

37. Construct the four antisymmetrized wavefunctions $\Psi(\vec{r}_1,\vec{r}_2,\sigma_1,\sigma_2) = \varphi(\vec{r}_1,\vec{r}_2)\chi(\sigma_1,\sigma_2)$ of a two-electron system, where σ_1 and σ_2 are the spin states of the two electrons and $\chi(\sigma_1,\sigma_2)$ is the total spin state. $\varphi(\vec{r}_1,\vec{r}_2)$ is the orbital part of the wavefunction. Assume that the two electrons occupy $\phi_a(\vec{r})$ and $\phi_b(\vec{r})$ two different orbitals.

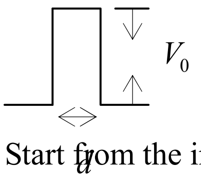
38. If a proton has a uniform charge distribution of radius R, the attractive potential between the electron and the proton will be $V(r) = -\dfrac{3e^2}{2R} + \dfrac{e^2r^2}{2R^3}$ for $r \leq R$ and $V(r) = -\dfrac{e^2}{r}$ for $r>R$. Calculate the first-order shift in the ground-state energy of hydrogen. You may assume $R < a_0$ (the Bohr radius).

39. For an attractive delta-function potential $V(x) = -aV_0\delta(x)$ use a Gaussian trial function to calculate the upper bound of E_0, the ground-state energy.

40. When a particle is scattered from a square well potential of depth V_0 and radius r_0, show that the s-wave phase shift is $\delta_0 = -kr_0 + \tan^{-1}\left(\dfrac{k}{k'}\tan k' r_0\right)$, where k and k' are the wave numbers inside and outside the well, respectively.

41. Using Taylor series expansion to find the operator $\hat{T}\vec{a}$ of a parallel displacement over a finite vector $\vec{a}$, defined by $\hat{T}\vec{a}\Psi(\vec{r}) = \Psi(\vec{r}+\vec{a})$, in terms of the momentum operator $\hat{p}$.

42. A particle of energy E is incident upon a potential barrier of height V_0 and width d (Fig.1). Calculate the transmission probability of this particle through the barrier as a function of E, where $E > V_0$. And locate maxima and minimum in the transmission probability.



43. Start from the infinitesimal time evolution operator

$$\cup(x,\varepsilon,x',o) = \left(\frac{m}{2\pi\hbar i\varepsilon}\right)^{1/2} \exp\left\{\frac{i}{\hbar}\left[\frac{m(x-x')^2}{2\varepsilon} - \varepsilon V\left(\frac{x+x'}{2},0\right)\right]\right\}$$

given by the path integral to prove that $\Psi(x,\varepsilon)$ satisfies the Schrödinger equation

$$\Psi(x,\varepsilon) - \Psi(x,0) = \frac{-i\varepsilon}{\hbar}\left[\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,0)\right]\Psi(x,0)$$

44. Calculate $<\ell' m'|[L_+, L_-]|\ell m>$, where $L_\pm = L_x \pm iL_y$ and

$L_z|\ell,m>= m\hbar|\ell,m>$, $L^2|\ell,m>= \ell(\ell+1)\hbar^2|\ell,m>$.

45. A spinor $\begin{pmatrix} a \\ b \end{pmatrix}$ is rotated by 30° with respect to the rotating axis

$\hat{n} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{k}$. What is the state after the rotation?

46. In the representation of the eigenstates of S_z, which is the z component of the spin operator $\vec{S}$ with S=$\frac{1}{2}$, S_z is written as $S_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. In the same representation, another spin operator $\hat{\Lambda}$ is defined by

$$\hat{\Lambda} = \frac{\hbar}{2}\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{\sqrt{2}}{4} + i\frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} - i\frac{\sqrt{2}}{4} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

Initially, we have a state which is an eigenstate of S_z with an eigenvalue of $+\frac{\hbar}{2}$. Now we make a measurement of $\hat{\Lambda}$.

(a) What's the probability of getting the $+\frac{\hbar}{2}$ eigenstate of S_z after $\hat{\Lambda}$ measurement?

(b) If we make a measurement of S_z right after $\hat{\Lambda}$, what's the probability of finding the $-\frac{\hbar}{2}$ eigenstate of S_z?

47. A free particle is restricted to move in $0 \le x \le L$. If the wavefunction of this particle is $\Psi(x) = \sin\frac{\pi x}{L} + 2\sin\frac{3\pi x}{L}$,

calculate (a) $\langle \hat{x} \rangle$, $\langle \hat{H} \rangle$ and (c) the probability of finding the particle in the region $0 \le x \le \frac{L}{2}$.

48. A beam of particles with a mass of m is scattered by a square well potential $V(r) = -V_0\theta(r_0 - r)$. Show that the differential cross section is

$$\frac{d\delta}{d\Omega} = 4r_0^{\,2}\left(\frac{mV_0r_0^{\,2}}{\hbar^2}\right)^2 \frac{(\sin qr_0 - qr_0\cos qr_0)^2}{(qr_0)^6},$$

where $\theta(r_0 - r)$ is the step function and $q = \left|\vec{K}_f - \vec{K}_i\right|$, $\vec{K}_i$ and $\vec{K}_i$ are the incident and scattered wavevectors of the particle, respectively.

49. A system has a Hamiltonian of $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}k\hat{x}^2 + \lambda\hat{x}^3$, where k and λ are constants. Which symmetries, translation, parity, time translation and time reversal, do the system have? And why? What conserved quantities do the symmetries correspond to?

50.. Calculate expectation values $\langle 1|\hat{x}|1\rangle$, $\langle 2|\hat{x}|2\rangle$, $\langle 1|\hat{p}|1\rangle$ and $\langle 2|\hat{p}|2\rangle$ for a simple harmonic oscillator, where $|1\rangle$ and $|2\rangle$ are eigenstates of $\hat{H}$ with eigenvalues $\frac{3}{2}\hbar\omega$ and $\frac{5}{2}\hbar\omega$, respectively.

51. The creation and annihilation operators $\hat{a}^+$ and $\hat{a}$ for fermions satisfy relations $\hat{a}\hat{a}^+ + \hat{a}^+\hat{a} = \hat{I}$ ($\hat{I}$: Identity operator), $\hat{a}\hat{a} = 0$ and $\hat{a}^+\hat{a}^+ = 0$.

(a) Show that the number operator $\hat{N} = \hat{a}^+\hat{a}$ satisfies $\hat{a}^+\hat{N} = (\hat{I} - \hat{N})\hat{a}^+$ and $\hat{a}\hat{N} = (\hat{I} - \hat{N})\hat{a}$.

(b) Show that the eigenvalues of $\hat{N}$ are 0 and 1.