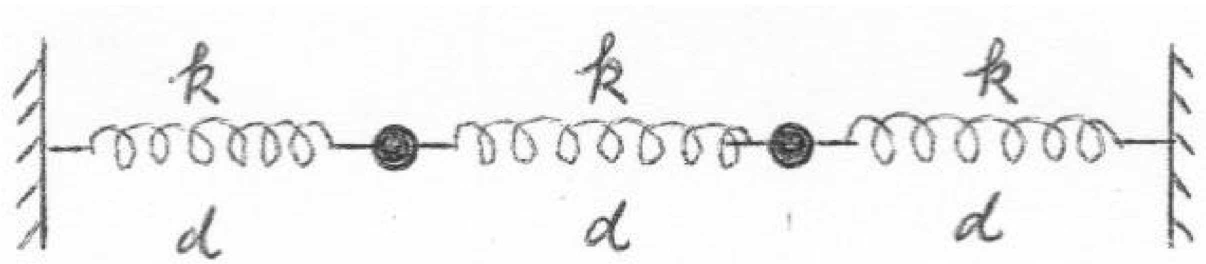
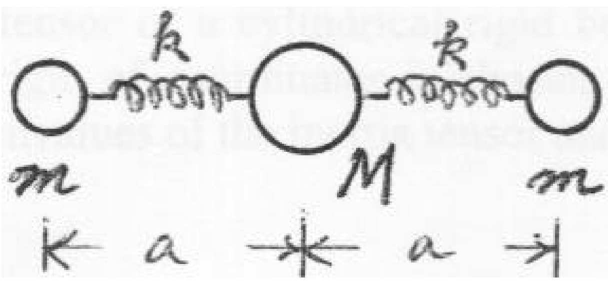
古典力學

1. An irregular-shaped object falls from the top of a very high tower to the ground. The frictional force exerted by air on this object is proportional to the square of its speed. Using the Lagrange's equation to find the equation of motion of this object. Obtain the velocity as a function of time and the terminal velocity of this object.
2. A solid uniform sphere rolls without slipping down a slope with an inclination angle of f. Using the method of Lagrange undetermined multipliers to find the equations of motion of this sphere. Then solve for the position and orientation of this sphere as functions of time. 「The moment of inertia of a solid uniform sphere rotating about an axis passing through its center is $I=(2/5)MR^2$,where M and R are its total mass and radius, respectively.」
3. Using the orthogonality property of the rotational transformation matrix to prove the Euler's theorem that the real orthogonal matrix specifying the physical motion of a rigid body with one point fixed always has the eigenvalues, +1, e^{if} and e^{-if}, where f is the rotational angle.
4. Two mass points move in one dimension at the junction of three springs, as shown in the figure. The springs all have unstretched lengths equal to d, and the force constants and masses are shown.



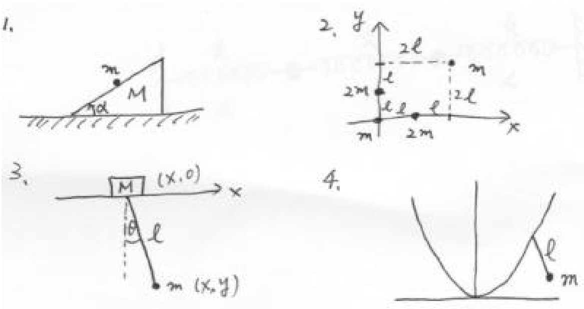
5. Using the Lagrange's equations to obtain the equations of motion of a two-body system under gravitational attraction.
6. Find the eigenfrequencies and eigenmodes of the small amplitude one dimensional vibrations of a three-body system shown in the figure.



7. Use the Hamilton's equations to obtain the equations of motion of two objects with masses of m_1 and m_2 connected by a spring with a spring constant of k.

8. Find the inertia coefficients, I_{jk}, of a uniform cubic rigid body with a width of a and a density of r . I_{jk}, is defined by $I_{jk} = \int_n r(\vec{r})(r^2 d_{jk} - x_j x_k) dV$,where j, k = 1,2, and 3. Choose the center of mass as the origin. Using these inertia coefficients to describe the angular momentum and kinetic energy of this rigid body.
9. Using the Lagrange equations to derive the equations of motion of a two-body system. These two bodies attract each other with a central force of $F=km_1m_2/r^3$, where m_1 and m_2 are their masses, r is the distance between them, and k is the proportional constant.
10. Derive the inertial tensor of a cylindrical rigid body with a radius of R and a length of L. The origin of coordinates is chosen at the center of this cylinder. Then obtain the eigenvaluses of the inertia tensor and the principal exes.
11. A linear mass-spring system has a mass M located at the center and two masses m located at both sides of M. Each mass m is connected to mass M by a spring with a spring constant k. Find the mass and kinetic energy matrices of this system. Then obtain the frequencies of free vibration and the corresponding normal coordinates.
12. Write down the canonical equations of Hamilton.
13. A particle of mass m slides without friction on a wedge of angle a and mass M that can move without friction on a smooth horizontal surface, as shown in the figure. Treating the constraint of the particle on the wedge by the method of Lagrange multipliers, find the equations of motion for the particle and wedge. Also obtain an expression for the forces of constraint. Calculate the work done in time t by the forces of constraint acting on the particle and on the wedge. What are the constants of motion for the system? Contrast the results with the situation when the wedge is fixed.
14. Four points masses are arranged (rigidly)as shown in the figure.
    (a) Construct the moment of inertia tensor.
    (b) Find the principal moments of inertia.
    (c) Find the direction of the principal axis.
    (d) If the masses rotate with constant angular velocity w about an axis that lies in y-z plane and passing through the origin making equal angles with y and z axes, find the direction (relative to the body axes) and the magnitude of the angular momentum vector.
15. Consider small oscillation problem for the following system : A simple pendulum of mass m is connected with a massless rod (length 1) to a block (mass M) which is confined to move along X-axis only. The pendulum moves in the plane of X-axis and vertical axis.
    (a) Write down the expressions for both T and V, and find the normal frequencies of the system.
    (b) Find the eigenvectors corresponding to the eigenfrequencies and construct the transformation matrix that rotates the original coordinate system to the normal coordinate system.
    (c) Find the normal modes of the system

16. The point of suspension of a simple pendulum of length 1 and mass m is constrained to move on a parabola $z = ax^2$ in the vertical plane. Derive a Hamiltonian governing the motion of the pendulum and its point of suspension. Obtain the Hamilton's equations of motion.



1. Consider a pendulum of length ℓ and a bob of mass m at its end moving through oil with q decreasing. The massive bob undergoes small oscillations, but the oil retards the bob's motion with a resistive force proportional to the speed with $F_{res} = 2m\sqrt{g/\ell}(\ell\dot{q})$. The bob is initially pulled back at t = 0 with $q = a, \dot{q} = 0$. Find the angular displacement q and velocity $\dot{q}$ as a function of time.
2. A simple pendulum placed in a railroad car that has a constant acceleration a in the x-direction. Find the Lagrange's equation of motion and the frequency of small oscillation of the simple pendulum.
3. A sphere of radius r is constrained to roll without slipping on the lower half of the inner surface of a hollow cylinder of inside radius R. Determine the Lagrangian function, the equation of constraint, and Lagrange's equations of motion. Find the frequency of small oscillation.
4. A pendulum consists of a mass m suspended by a massless spring with unextended length b and spring constant k. (a) Find Lagrange's equations of motion. (b) Using the Lagrangian method find the equations of motion. (c) Determine the Hamiltonian and Hamilton's equations of motion. (d) What is the period of small oscillations?
5. Find (a) the force law, (b) r(t), (c) $q(t)$ and (d) the total energy for a central-force field that allows a particle to move in a logarithmic spiral orbit given by $r = ke^{aq}$, where k and a are constants.
6. A particle of mass m_1, momentum p_1, collides elastically with a particle of mass m_2 which is initially at rest. Calculate the maximum possible momentum of particle m_2 after the collision in the laboratory system. (Use relativistic correct formulas.) Apply your result to the case of a proton having a momentum equal to its rest energy/c colliding with an electron at rest. Given a numerical value for the maximum momentum of the electron after the collision in MeV/c.
7. A moving proton collides with another proton that is initially at rest. Calculate the minimum kinetic energy that the moving proton must have in order to make possible the reaction $p + p = p + p + p + \overline{p}$. The antiproton ($\overline{p}$) has the same mass as the proton. Express your result as the ratio of the kinetic energy to the rest energy of the proton. What is the approximate numerical value for the kinetic energy in MeV?
8. A uniform thin rod of length L, mass m, is lying on the usual smooth horizontal table. A

horizontal impulse **J** is suddenly applied perpendicular to the rod at one end. (a) How far does the rod travel during the time it takes to make one complete revolution? (b) What are the translational, rotational, and total kinetic energies of the rod after the impulse?

9. (a) A rigid rod of length a and mass m is suspended by equal massless threads of L fastened by its ends. While hanging at rest, it receives a small impulse **J** at one end in a direction perpendicular to the rod and to the thread. Calculate the normal frequencies and the amplitudes of the associated normal modes in the subsequent motion. (b) If the rod is hanging at rest and suddenly one of the strings is cut, what is the tension in the other string immediately thereafter?
10. A uniform bar of length Land mass m is supported at its ends by identical springs with elastic constant k (Fig.1). Motion is initiated by depressing one end by a small distance a and releasing it from rest. Solve the problem of the motion, identifying normal modes and frequencies.
11. A particle of mass m moves under the action of a central force whose potential is $V(r)=kr^3$ (k>0). (a) For what energy and angular momentum will the orbit be a circle of radius a about the origin? (b) What is the period of this circular motion? (c) If the particle is slightly disturbed from this circular motion, what will be the period of small radial oscillations about r=a ?
12. An unstable particle of rest energy 1000 MeV decays into a m-meson ($m_0c^2 = 100 MeV$) and a neutrino with a mean life, when at rest, of 10^{-8} sec. (a) Calculate the mean decay distance when the particle has a momentum of 1000 MeV/c. (b)What is the energy of the m-meson if it is emitted at an angle 15^0?
13. A football in the shape of an ellipsoid of revolution has moments of inertia about its principal axes of I_1, I_2 and I_3. When thrown it is accidentally given an angular velocity w which makes an angle $q(<90^0)$ with long axis of the football. (a) What is the magnitude and direction (with respect to the long axis of the football) of the angular momentum? (b) What is the rate of precession (neglet air friction and gravity).
14. Given a vector V and a tensor T, V = $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, T = $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ both referred to a two-dimensional Cartesian coordinate system x_1, x_2. Compute the components of the vector and the tensor referred to the new coordinate system obtained by a rotation of the old coordinate system by 90^0 in a positive sense about the $x_1 \times x_2$ axis.
15. Two particles moving under the influence of their mutual gravitational force describe circular orbits about one another with a period t. If they are suddenly stopped in their orbits and allowed to gravitate toward each other, show that they will collide after a time $t/4\sqrt{2}$.
16. Consider a rope of mass per unit length l and length L suspended just above a table. If the rope is released from rest at the tope, find the force on the table when a length x of the rope r has dropped to the table.
17. Derive the Rutherford scattering formula. (The scattering of charged particles in a Coulomb electrostatic field)
18. Find the horizontal deflection from the plumb line caused by the Coriolis force acting on a particle falling freely in the Earth's gravitational field from a height h above the Earth's surface.

19. (a) Calculate the inertia tensor of a homogeneous cube of density r, mass M, and side of length b. Let one corner be at the origin, and let three adjacent edges lie along the coordinate axes. (b) Find the principal moments of inertia and the principal axes for cube in (a).
20. Determine the eigenfrequencies and describe the normal mode motion of a symmetrical linear tri-atomic molecule (fig.2). The central atom has mass M, and the symmetrical atoms have masses m. Both longitudinal and transverse vibrations are possible.
21. Consider three identical pendula suspended from a slightly yielding support. Because the support is not rigid, a coupling occurs between the pendula, and energy can be transferred from one pendulum to the other(Fig.3). Find the eigenfrequencies and eigenvectors and describe the normal mode motion.
22. Calculate the differential cross section $s(q)$ and the total cross section s_t for the elastic scattering of a particle from an impenetrable sphere; the potential is given by $U(r)=\begin{Bmatrix} 0, r>a \\ \infty, r<a \end{Bmatrix}$.
23. Calculate the differential cross section $s(q)$ for the scattering of charged particles in a Coulomb field; the potential is U(r) = k/r.
24. A fixed force center scatters a particle of mass m according to the force law $F(r)=k/r^3$. If the initial velocity of the particle is u_0, show that the differential scattering cross section is $s(q)=\dfrac{kp^2(p-q)}{mu_0^2q^2(2p-q)^2\sin q}$.
25. Show that the path of a particle moving under the influence of a central force whose magnitude is inversely proportional to the square of the distance between the particle and the force center can be written as $\dfrac{a}{r}=1+e\cos q$, where the quantity e is called the eccentricity, and $2a$ is termed the latus rectum of the orbit.
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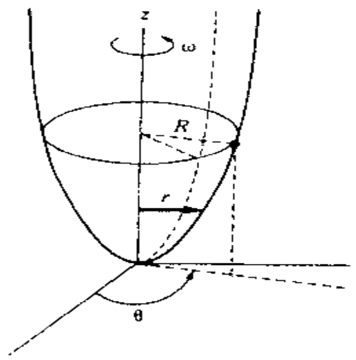
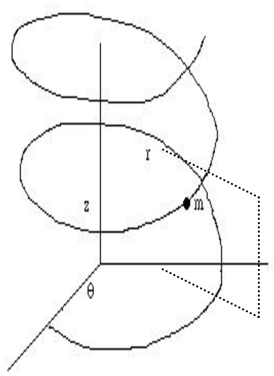
1. Find the force of attraction of a thin spherical shell of radius **a** on a particle P of mass **m** at a distance r > a from its center.
2. Find the dimensions of the parallelepiped of maximum volume circumscribed by an ellipsoid with semiaxes a,b,c.
3. Consider the surface generated by revolving a line connecting two fixed points (X_1, Y_1) and (X_2, Y_2) about an axis coplanar with the two points. Find the equation of the line connecting the points such that the surface area generated by the revolution is a minimum.

   【$\int \frac{a}{\sqrt{x^2 - a^2}} dx = a\cosh^{-1}\frac{x}{a} + b$】
4. A bead slides along a smooth wire bent in the shape of a parabola $\mathbf{z = cr^2}$. The bead rotates in a circle of radius **R** when the wire is rotating about its vertical symmetry axis with angular velocity **?** . Find the value of **c**.



5. A double pendulum consists of two simple pendula, with one pendulum suspended from the bob of the other. If the two pendula have equal lengths ℓ and have bobs of equal mass **m** and if both pendula are confined to move in the same plane, find Lagrange's equations of motion for the system. Do not assume small angles.
6. A particle of mass m moves under the influence of gravity along the spiral **z = K?**, **r** = constant, where **K** is a constant and **z** is vertical. Obtain the Hamiltonian equtions of motion.



7. A pendulum consists of a mass m suspended by a massless spring with unextended length **b** and spring constant **K**. If the pendulum's point of support rises vertically with constant acceleration **a**, find the Hamilton's equations of motion.

8. A disk rolls without slipping down an inclined plane with the inclination angle **f** . Discuss the

motion of this disk and find the force of constraint by using the method of undetermined multipliers. (The moment of inertia of the disc about a symmetrical axis is I = $\frac{1}{2}MR^2$, where M and R are its mass and radius, respectively.)

9. A mass **m** moves in a circular orbit of radius $\mathbf{r_0}$ under the influence of a central force whose potential is $-km / r^n$. Find the condition that the orbit is stable under small oscillations.
10. A string connects two particles with equal mass **m** through a hole in a horizontal frictionless table. One particle moves on the surface of the table, the other hangs vertically. At what speed $\mathbf{V_0}$ does the particle on the table move in a perfect circle of radius $\mathbf{r_0}$? If the hanging particle is raised a small distance e about its equilibrium position and released, find the subsequent motion.
11. A particle of mass **m** is observed to move with angular momentum ℓ in a spiral orbit given by the equation **r = k?** , where **k** is a constant. Find the total energy of the orbit.
12. A particle of mass **m** is moving in a circle of radius $\mathbf{r_0}$ under the action of an attractive force **F** $= -\frac{1}{r^2}e^{-\frac{r}{a}}$, where **a > 0**. Find the condition that the circular motion has a stable orbit at radius $\mathbf{r_0}$.
13. Two particles moving under the influence of their mutual gravitational force describe circular orbits about one another with a period t **.** If they are suddenly stopped in their orbits and allowed to gravitate toward each other. Find the time that they will collide after stopping circular orbits.

【 $\int\frac{x^2dx}{\sqrt{a^2-x^2}}=-\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}$ 】

14. Calculate the differential cross section $s(q)$ and the total cross section s_t for the elastic scattering of a particle from a impenetrable sphere; the potential is given by

$$U(r)=\begin{cases}0, & r>a\\ \infty, & r<a\end{cases} \quad , 【\int\frac{dx}{x\sqrt{ax^2+bx+c}}=\frac{1}{\sqrt{-c}}\sin^{-1}\left(\frac{bx+2c}{|x|\sqrt{b^2-4ac}}\right) \text{ for } c<0,\ b^2>4ac】$$

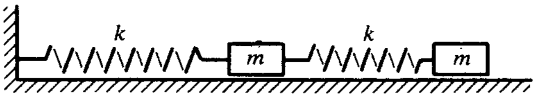
15. A proton (mass **m**) of kinetic energy $\mathbf{T_0}$ collides with a helium nucleus (mass **4m**) at rest. Find the recoil angle of the helium if the scattered angle **?** of proton is **45°** and the energy loss during the inelastic collision is $\frac{T_0}{6}$.
16. Find the horizontal deflection from the plumb line caused by the Coriolis force acting on a particle falling freely in the Earth's gravitational field from a height **h** above Earth's suface.

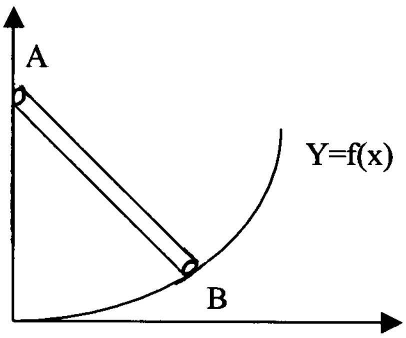
17. Consider a homogeneous cube of density **?** , mass **M**, and side **L**. For a coordinate system with its origin at one corner of the cube and three axes directed along the edges, evaluate the principal axes and their associated moment of inertia.
18. One end point A of a rigid rod of length ℓ leans against a perfectly smooth vertical wall, while the other end point B is supported by a perfectly smooth curved surface, given by the equation Y = f(x) (in the coordinate system shown in the figure).

And f(0) = 0.

(a)What should f(x) be in order that the rod be in equilibrium for all positions A and B?

(b)Does such a curve have a special name?

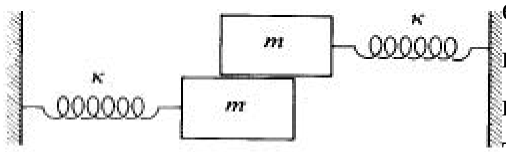




19. A homogeneous disk of radius **R** and mass **M** rolls without slipping on a horizontal surface and is attracted to a point a distance **d** below the plane. If the force of attraction is proportional to the distance from the disk's center of mass to the force center, find the frequency of oscillations around the position of equilibrium.

20. Two identical harmonic oscillators are coupled via their sliding friction force against each other as shown. The frictional force is proportional to their instaneous relative velocity. Discuss the coupled oscillations of the system.

21. Determine the eigenfrequencies and describe the normal mode motion for two pendula of equal lengths **b** and equal masses **m** connected by a spring of force constant **K**. The spring is unstretched in the equilibrium position.



22. Determine the ratio of normal frequencies of the longitudinal vibration for a symmetrical linear triatomic molecule CO_2.

23. Three identical masses **m** are connected by identical springs of stiffness constant **K** and placed on a fixed circular loop in space. Each spring is unstretched when the system is at equilibrium and the masses are constrained to oscillate in the same circular loop. Calculate the normal frequencies of this system.

24. Consider the frictionless system of two equal masses **m** and two identical springs (of constant **K**) as shown. Find the normal mode frequencies and describe the motions of the system for each of the normal modes.

25. Given three equal masses $\mathbf{m_1=m_2=m_3=m}$ connected by springs, all of the same spring constant **K** and of the same length **a**, forming an equilateral triangle. For small oscillations from the rest positions, find the frequency of the simplest, radially symmetric mode of oscillation of this system.